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Almost Semi-Equivelar Maps on Torus and Klein Bottle

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Abstract: A map M is called an almost semi-equivelar if all the vertices of M have same type face-cycle except one. The maps on the surfaces of square pyramid and the pentagonal pyramid (2 out of 92 Johnson solids) provide almost semi-equivelar maps on the sphere. In this paper, for the first time, we study and classify an almost semi-equivelar map on close surfaces, other than sphere, particularly on torus and Klein bottle on at most 15 vertices.

Key Words: Semi-equivelar map, almost semi-equivelar map, torus, Klein bottle.

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§1. Introduction

For the topological graph theory related terms, we refer to [7]. A surface (closed) F is a connected, compact 2-dimensional manifold without boundary. An embedding of a connected, simple graph G into F is called a map M on F if the closure of each connected component of $F \setminus G$ is a p-gonal 2-disk D_p, also called face of the map, such that the non-empty intersection of any two faces is either a vertex or an edge, see [1]. The vertices and edges of the underlying graph G are called the vertices and edges of M. In this paper, we deal with the maps on torus and Klein bottle. Let M_1 and M_2 be two maps with vertex sets $V(M_1)$ and $V(M_2)$ respectively. Then M_1 is isomorphic to M_2, denoted as $M_1 \cong M_2$, if there is a bijective map $f : V(M_1) \to V(M_2)$ that preserves the incidence of edges and faces.

In a map M, a cycle of consecutive faces $(D_{p_1}, \cdots D_{p_k})$ around a vertex v such that $D_{p_1} \cap D_{p_2}, \ldots, D_{p_k} \cap D_{p_1}$ are edges, is called the face-cycle of v. A map is called semi-equivelar of type $(D_{p_1}. \cdots .D_{p_k})$ if the face-cycle of each vertex is same and of the type $(D_{p_1}, D_{p_2}, \cdots D_{p_k})$ up to a cyclic permutation. The well known five Platonic solids and thirteen Archimedean solids provide all possible types semi-equivelar maps on the sphere, *i.e.*, the surface of Euler characteristic 2. Such maps have been studied extensively by many researchers for the surfaces other than the sphere. For a recent progress on such maps for the surfaces of Euler characteristic 0, i.e., on torus and Klein bottle, see ([4], [5], [3], [13], [11]), for the surface of Euler characteristic -1, see ([13], [14]) and for the surfaces of Euler characteristic -2, see ([6], [10], [9]).

A map M is called an almost semi-equivelar map, briefly ASEM, if all the vertices have

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same face-cycle except one vertex. If M has one vertex with face-cycle (f_1) and remaining vertices with face-cycle (f_2), we say that M is ASEM of the type $[(f_1)_1 : (f_2)]$. The surfaces of the square pyramid and pentagonal pyramid (2 of the 92 Johnson solids) provide ASEMs of types $[(3,3,3,3)_1 : (3,3,4)]$ and $[(3,3,3,3,3)_1 : (3,3,4)]$ respectively on sphere, see [8]. In this paper, we study and classify ASEMs on torus and Klein bottle.

The article is organized as follows:, we describe almost semi-equivelar map of the type $[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ for the surfaces of Euler characteristic $\chi = 0$ and give a methodology to enumerate this type maps in Section 2. In Section 3, we present the results obtained during the enumeration. In Section 4, we illustrate the methodology and prove the result. We conclude the article by presenting some observations related to such almost semi-equivelar maps.

§2. Methodology

Let M be an ASEM of the type $[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$. Let v be a vertex in M with the face-cycle $(D_{p_1}, D_{p_2}, \cdots, D_{p_k})$. Then the union of these disks, i.e., $\cup_{i=1}^{k} D_{p_i}$ is a 2-disk D_n with the boundary cycle C_n, where $n = (p_1 + p_2 + \ldots + p_k) - 2k$. Let us call this cycle C_n as the link of v and denote it as $\mathrm{lk}(v)$. Thus $\mathrm{lk}(v)$ is a six or seven length cycle if the face-cycle of v is $(3,3,3,3,3,3)$ or $(3,3,4,3,4)$ respectively. We use the following notations to represent these links.

The notation $\mathrm{lk}(v) = C_6(v_1, v_2, v_3, v_4, v_5, v_6)$ means that the face-cycle of v is $(3,3,3,3,3,3)$, i.e., triangular faces $[v, v_1, v_2]$, $[v, v_2, v_3]$, $[v, v_3, v_4]$, $[v, v_4, v_5]$, $[v, v_5, v_6]$ and $[v, v_6, v_1]$ are incident at v.

The notation $\mathrm{lk}(v) = C_7(v_1, v_2, v_3, \boldsymbol{v}_4, v_5, v_6, \boldsymbol{v}_7)$ means that the face-cycle of v is $(3,3,4,3,4)$, i.e., triangular faces $[v, v_1, v_2]$, $[v, v_2, v_3]$, $[v, v_5, v_6]$ and quadrangular faces $[v, v_3, v_4, v_5]$, $[v, v_6, v_7, v_1]$ are incident at v. Note that, here bold symbols $\boldsymbol{v}_i$ shows that $\boldsymbol{v}_i$ is not adjacent to v.

Let $V(M) = \{u, v_1, \ldots, v_n\}$ be the vertex set of M. Here the vertex u has the face-cycle $(3,3,3,3,3,3)$ and the remaining have $(3,3,4,3,4)$. Let $\mathrm{lk}(u) = C_6(v_1, v_2, v_3, v_4, v_5, v_6)$. Now, we use the following steps to enumerate the ASEMs.

Step 1. Without loss of generality, choose v_1 (one can choose any v_i for $1 \le i \le 6$) such that $\mathrm{lk}(v_1) = C_7(v_2, u, v_6, \boldsymbol{\alpha}_1, \alpha_2, \alpha_3, \boldsymbol{\alpha}_4)$, where α_i, for $1 \le i \le 4$, is some $v_j \in V$.

Step 2. For each choice of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in $V(M) \times V(M) \times V(M) \times V(M)$, we construct $\mathrm{lk}(v_1)$.

Step 3. For each possibility of $\mathrm{lk}(v_1)$ obtained from Step 2, we repeat Step 1 and Step 2 until we do not get links of remaining vertices.

Step 4. Define an isomorphism (if possible) between the maps, which will lead to the enumeration of the maps.

Applying the above methodology for $|V| \le 15$, we obtain the following result.

§3. Result

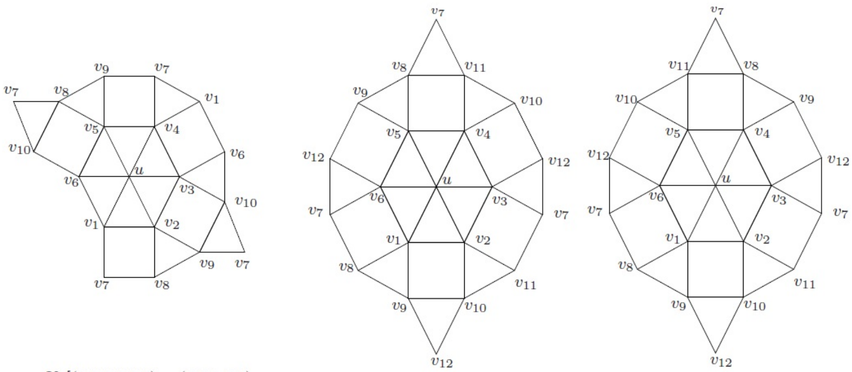
Theorem 3.1 *Let M be an ASEM of type $[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ with the vertex set*

$V(M)$. Then

(1) *M does not exist for $|V(M)| \leq 9$;*

(2) *A unique M exists for $|V(M)| \leq 12$, this is $K_1[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ on Kelin bottle with 11 vertices;*

(3) *There exist exactly three such maps for $|V(M)| \leq 15$ on the surfaces of Euler characteristic $\chi = 0$. The other two maps are $K_2[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ and $T_1[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ on torus and Klein bottle, respectively, with $|V(M)| = 13$.*



$K_1[(3.3.3.3.3.3)_1 : (3.3.4.3.4)$ $K_2[(3.3.3.3.3.3)_1 : (3.3.4.3.4)$ $T_1[(3.3.3.3.3.3)_1 : (3.3.4.3.4)$

Figure 3.1 ASEMs on Klein bottle

§4. Proof: Classification of ASEMs of Type $[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$

Let M be an ASEM of type $[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ with the vertex set $V(M) = V_{(3,3,3,3,3,3)} \cup V_{(3,3,4,3,4)}$, where $V_{(3,3,3,3,3,3)}$ and $V_{(3,3,4,3,4)}$ denote the sets of vertices with face-cycles $(3,3,3,3,3,3)$ and $(3,3,4,3,4)$ respectively. Thus for $|V| \leq 15$, we let $V_{(3,3,3,3,3,3)} = \{0\}$ and $V_{(3,3,4,3,4)} = \{1, 2, \cdots, n\}$, where $n \leq 14$.

Let $\mathrm{lk}(0) = C_6(1,2,3,4,5,6)$. This implies $\mathrm{lk}(1) = C_7(2,0,6,\boldsymbol{a},b,c,\boldsymbol{d})$, where $a,b,c,d \in V_{(3,3,4,3,4)}$. It is easy to see that $(a,b,c,d) \in \{(3,4,7,8),(4,3,7,8),(7,8,4,5),(7,8,5,4),(7,8,9,10)\}$. Observe that, $(3,4,7,8) \cong (7,8,4,5)$ and $(4,3,7,8) \cong (7,8,5,4)$ by the map $(2,6)(3,5)(7,8)$. Thus, we search for $(a,b,c,d) \in \{(4,3,7,8),(3,4,7,8),(7,8,\ 9,10)\}$.

Case 1. In the case $(a,b,c,d) = (4,3,7,8)$, $\mathrm{lk}(1) = C_7(2,0,6,\mathbf{4},3,7,\mathbf{8})$. This implies $\mathrm{lk}(3) = C_7(2,0,4,\ \mathbf{6},1,7,\boldsymbol{e})$ and we see that two distinct quadrangular faces share more than one vertex. This is not allowed. Thus $(a,b,c,d) \neq (4,3,\ 7,8)$

Case 2. In case $(a,b,c,d) = (3,4,7,8)$, we get $\mathrm{lk}(1) = C_7(2,0,6,\mathbf{3},4,7,\mathbf{8})$. Now considering $\mathrm{lk}(i)$, for $1 \leq i \leq 6$ and the fact that two distinct quadrangular faces share at most one vertex, we see easily that $\mathrm{lk}(4) = C_7(5,0,3,\mathbf{6},1,7,\mathbf{9})$. This implies $\mathrm{lk}(7) = C_7(8,10,9,\mathbf{5},4,1,\mathbf{2})$ and $\mathrm{lk}(2) = C_7(3,0,1,\mathbf{7},8,e,\boldsymbol{f})$. Observe that $(e,f) \in \{(9,10),(11,9),(11,10),(11,12)\}$.

Subcase 2.1. When $(e,f) = (11,i)$, for $i \in \{9,10\}$, then $\mathrm{lk}(2) = C_7(3,0,1,\mathbf{7},8,11,\boldsymbol{i})$ and $\mathrm{lk}(3) = C_7(2,0,4,\mathbf{1},6,i,\mathbf{11})$. This implies, $\deg(i) > 5$. Hence $(e,f) \neq (11,9)$ or $(11,10)$.

Subcase 2.2. When $(e, f) = (11, 12)$, then $\mathrm{lk}(2) = C_7(3, 0, 1, \mathbf{7}, 8, 11, \mathbf{12})$. This implies $\mathrm{lk}(3) = C_7(2, 0, 4, \mathbf{1}, 6, 12, \mathbf{11})$. Then $\mathrm{lk}(6) = C_7(5, 0, 1, \mathbf{4}, 3, 12, \mathbf{13})$, $\mathrm{lk}(5) = C_7(4, 0, 6, \mathbf{12}, 13, 9, \mathbf{7})$ and $\mathrm{lk}(9) = C_7(14, 13, 5, \mathbf{4}, 7, 10, \mathbf{15})$, $\mathrm{lk}(10) = C_7(8, 7, 9, \mathbf{14}, 15, j, \mathbf{k})$, now we see that j and k have no admissible values in $V_{(3,3,4,3,4)}$. Thus $(e, f) \neq (11, 12)$

Subcase 2.3 When $(e, f) = (9, 10)$ then successively we get $\mathrm{lk}(2) = C_7(3, 0, 1, \mathbf{7}, 8, 9, \mathbf{10})$, $\mathrm{lk}(3) = C_7(2, 0, 4, \mathbf{1}, 6, 10, \mathbf{9})$. This implies $\mathrm{lk}(6) = C_7(5, 0, 1, \mathbf{4}, 3, 10, \boldsymbol{g})$, where $g \in \{8, 11\}$. If $g = 11$, $\deg(10) > 5$, a contradiction. On the other hand when $g = 8$, completing successively, we get $\mathrm{lk}(6) = C_7(5, 0, 1, \mathbf{4}, 3, 10, \mathbf{8})$, $\mathrm{lk}(5) = C_7(4, 0, 6, \mathbf{10}, 8, 9, \mathbf{7})$, $\mathrm{lk}(9) = C_7(2, 8, 5, \mathbf{4}, 7, 10, \mathbf{3})$, $\mathrm{lk}(8) = C_7(2, 8, 5, \mathbf{6}, 10, 7, \mathbf{1})$ and $\mathrm{lk}(10) = C_7(8, 7, 9, \mathbf{2}, 3, 6, \mathbf{5})$. This gives $M \cong K_1(3^6 : 3^2, 4, 3, 4)$ by the map $0 \to u$, $i \to v_i$ for $1 \leq i \leq 10$.

Case 3. In case $(a, b, c, d) = (7, 8, 9, 10)$, $\mathrm{lk}(1) = C_7(2, 0, 6, \mathbf{7}, 8, 9, \mathbf{10})$. This implies $\mathrm{lk}(2) = C_7(3, 0, 1, \mathbf{9}, 10, e, \boldsymbol{f})$. It is easy to see that $(e, f) \in \{(5, 6), (7, 11), (11, 7), (11, 8), (11, 12)\}$. Observe that, $(7, 11) \cong (11, 8)$ by the map $(1, 2)(3, 6)(4, 5)(7, 8, 11)(9, 10)$. Therefore, we search for $(e, f) \in \{(5, 6), (11, 7), (11, 8), (11, 12)\}$.

Subcase 3.1. If $(e, f) = (5, 6)$, $\mathrm{lk}(2) = C_7(3, 0, 1, \mathbf{9}, 10, 5, \mathbf{6})$ and $\mathrm{lk}(6) = C_7(1, 0, 5, \mathbf{2}, 3, 7, \mathbf{8})$. This implies $\mathrm{lk}(5) = C_7(4, 0, 6, \mathbf{3}, 2, 10, \boldsymbol{g})$, where $g \in \{7, 8, 11\}$. If $g = 7$ successively we get $\mathrm{lk}(5) = C_7(4, 0, 6, \mathbf{3}, 2, 10, \mathbf{7})$, $\mathrm{lk}(10) = C_7(7, 11, 9, \mathbf{1}, 2, 5, \mathbf{4})$, $\mathrm{lk}(7) = C_7(8, 11, 10, \mathbf{5}, 4, 6, \mathbf{1})$. This implies $C_5(0, 1, 8, 7, 5) \subseteq \mathrm{lk}(6)$. If $g = 11$, $\mathrm{lk}(5) = C_7(4, 0, 6, \mathbf{3}, 2, 10, \mathbf{11})$, successively, we get $\mathrm{lk}(10) = C_7(9, 12, 11, \mathbf{4}, 5, 2, \mathbf{1})$, $\mathrm{lk}(4) = C_7(3, 0, 5, \mathbf{10}, 11, 13, \mathbf{7})$, $\mathrm{lk}(3) = C_7(2, 0, 4, \mathbf{13}, 7, 6, \mathbf{5})$ and $\mathrm{lk}(7) = C_7(8, 14, 13, \mathbf{4}, 3, 6, \mathbf{1})$. Now a small computation shows that $\mathrm{lk}(8)$ can not be completed for the given $V_{(3,3,4,3,4)}$. If $g = 8$, $\mathrm{lk}(5) = C_7(4, 0, 6, \mathbf{3}, 2, 10, \mathbf{8})$. This implies $\mathrm{lk}(10) = C_7(8, h, 9, \mathbf{1}, 2, 5, \mathbf{4})$. Clearly $h = 7$, completing successively, we get $\mathrm{lk}(8) = C_7(1, 9, 4, \mathbf{5}, 10, 7, \mathbf{6})$, $\mathrm{lk}(3) = C_7(2, 0, 4, \mathbf{9}, 7, 6, \mathbf{5})$, $\mathrm{lk}(9) = C_7(1, 8, 4, \mathbf{3}, 7, 10, \mathbf{2})$, $\mathrm{lk}(7) = C_7(8, 10, 9, \mathbf{4}, 3, 6, \mathbf{1})$, $\mathrm{lk}(4) = C_7(3, 0, 5, \mathbf{10}, 8, 9, \mathbf{7})$. This gives a map $M \cong K_1(3^6 : 3^2, 4, 3, 4)$ by the map $0 \mapsto u$, $1 \mapsto v_2$, $2 \mapsto v_1$, $3 \mapsto v_6$, $4 \mapsto v_5$, $5 \mapsto v_4$, $6 \mapsto v_3$, $7 \mapsto v_{10}$, $8 \mapsto v_9$, $9 \mapsto v_8$, $10 \mapsto v_7$.

Subcase 3.2. When $(e, f) = (11, 7)$ then $\mathrm{lk}(2) = C_7(1, 0, 3, \mathbf{7}, 11, 10, \mathbf{9})$. This implies $\mathrm{lk}(3) = C_7(4, 0, 2, \mathbf{11}, 7, g, \boldsymbol{h})$, where $(g, h) \in \{(8, 12), (12, 9), (12, 10), (12, 13)\}$.

If $(g, h) = (8, 12)$, $\mathrm{lk}(8) = C_7(1, 9, 12, \mathbf{4}, 3, 7, \mathbf{6})$. This implies $\mathrm{lk}(9) = C_7(12, 8, 1, \mathbf{2}, 10, i, \boldsymbol{j})$. Observe that, $(i, j) \in \{(5, 6), (13, 11), (13, 14)\}$. If $(i, j) = (5, 6)$, considering $\mathrm{lk}(6)$, we see that the degree of 7 is more than 5. If $(i, j) = (13, 11)$ or $(13, 14)$, we see $\mathrm{lk}(12)$ can not be completed. Thus $(g, h) \neq (8, 12)$

If $(g, h) = (12, 9)$, $\mathrm{lk}(4) = C_7(5, 0, 3, \mathbf{12}, 9, i, \boldsymbol{j})$, where $(i, j) \in \{(8, 11), (10, 13)\}$. In case $(i, j) = (10, 13)$, $\mathrm{lk}(10) = C_7(2, 11, 13, \mathbf{5}, 4, 9, \mathbf{1})$ and $\mathrm{lk}(11) = C_7(13, 10, 2, \mathbf{3}, 7, k, \boldsymbol{l})$, where $(k, l) \in \{(6, 5), (8, 12), (8, 14)\}$. When $(k, l) = (6, 5)$ then $C_5(0, 4, 10, 13, 11, 6) \subseteq \mathrm{lk}(5)$. When $(k, l) = (8, 12)$ then, successively, considering $\mathrm{lk}(8)$, $\mathrm{lk}(9)$, $\mathrm{lk}(12)$, we see that $\deg(7) > 5$. When $(k, l) = (8, 14)$ then considering $\mathrm{lk}(8)$, we see that $\deg(9) > 5$. On the other hand when $(i, j) = (8, 11)$, completing successively we get $\mathrm{lk}(8) = C_7(1, 9, 4, \mathbf{5}, 11, 7, \mathbf{6})$, $\mathrm{lk}(9) = C_7(1, 8, 4, \mathbf{3}, 12, 10, \mathbf{2})$, $\mathrm{lk}(11) = C_7(2, 10, 5, \mathbf{4}, 8, 7, \mathbf{3})$, $\mathrm{lk}(10) = C_7(2, 11, 5, \mathbf{6}, 12, 9, \mathbf{1})$, $\mathrm{lk}(5) = C_7(4, 0, 6, \mathbf{12}, 10, 11, \mathbf{8})$, $\mathrm{lk}(6) = C_7(1, 0, 5, \mathbf{10}, 12, 7, \mathbf{8})$, $\mathrm{lk}(7) = C_7(3, 12, 6, \mathbf{1}, 8, 11, \mathbf{2})$. This gives $M \cong T_1(3^6 : 3^2, 4, 3, 4)$ by the map $0 \mapsto u$, $i \mapsto v_i$ for $1 \leq i \leq 12$.

If $(g, h) = (12, 10)$, $\mathrm{lk}(3) = C_7(2, 0, 4, \mathbf{10}, 12, 7, \mathbf{11})$. This implies $\mathrm{lk}(10) = C_7(2, 11, 12, \mathbf{3},$

$4,9,\mathbf{1})$ or $\mathrm{lk}(10) = C_7(2,11,4,\mathbf{3},12,9,\mathbf{1})$. In the first case, *i.e.*, when $\mathrm{lk}(10) = C_7(2,11,12,\mathbf{3},4,9,\mathbf{1})$ we get $\mathrm{lk}(4) = C_7(3,0,5,\mathbf{13},9,10,\mathbf{12})$, $\mathrm{lk}(9) = C_7(1,8,13,\mathbf{5},4,10,\mathbf{2})$, now observe that, $\mathrm{lk}(7) = C_7(3,12,8,\mathbf{1},6,11,\mathbf{2})$ or $\mathrm{lk}(7) = C_7(3,12,6,\mathbf{1},8,11,\mathbf{2})$. When $\mathrm{lk}(7) = C_7(3,12,8,\mathbf{1},6,11,\mathbf{2})$ then, successively considering lk(8) and lk(12), we see $\deg(10) > 5$, while for $\mathrm{lk}(7) = C_7(3,12,6,\mathbf{1},8,11,\mathbf{2})$, considering lk(8), we see three quadrangular faces incident at 11. Thus $\mathrm{lk}(10) \neq C_7(2,11,12,\mathbf{3},4,9,\mathbf{1})$. On the other hand when $\mathrm{lk}(10) = C_7(2,11,4,\mathbf{3},12,9,\mathbf{1})$ then $\mathrm{lk}(4) = C_7(5,0,3,\mathbf{12},10,11,\boldsymbol{i})$, clearly $i = 8$, now completing successively we get $\mathrm{lk}(11) = C_7(2,10,4,\mathbf{5},8,7,\mathbf{3})$, $\mathrm{lk}(7) = C_7(3,12,6,\mathbf{1},8,11,\mathbf{2})$, $\mathrm{lk}(12) = C_7(3,7,6,\mathbf{5},9,10,\mathbf{4})$, $\mathrm{lk}(6) = C_7(1,0,5,\mathbf{9},12,7,\mathbf{8})$, $\mathrm{lk}(5) = C_7(4,0,6,\mathbf{12},9,8,\mathbf{11})$, $\mathrm{lk}(8) = C_7(1,9,5,\mathbf{4},11,7,\mathbf{6})$, $\mathrm{lk}(9) = C_7(1,8,5,\mathbf{6},12,10,\mathbf{2})$. This gives $M \cong K_2(3^6 : 3^2,4,3,4)$ by the map $0 \mapsto u$, $i \mapsto v_i$ for $1 \le i \le 12$.

When $(g,h) = (12,13)$, then $\mathrm{lk}(7) = C_7(3,12,6,\mathbf{1},8,11,\mathbf{2})$ or $\mathrm{lk}(7) = C_7(3,12,8,\mathbf{1},6,11,\mathbf{2})$. In the first case when $\mathrm{lk}(7) = C_7(3,12,6,\mathbf{1},8,11,\mathbf{2})$, we get $\mathrm{lk}(6) = C_7(5,0,1,\mathbf{8},7,12,\boldsymbol{i})$, where $i \in \{9,10,14\}$. If $i = 9$ and 10, considering lk(12), we see $\deg(9) > 5$ and $\mathrm{lk}(10) > 5$ respectively. If $i = 14$, $\mathrm{lk}(12) = C_7(3,7,6,\mathbf{5},14,13,\mathbf{4})$. This implies $\mathrm{lk}(5) = C_7(4,0,6,\mathbf{12},14,j,\boldsymbol{k})$ and $\mathrm{lk}(4) = C_7(5,0,3,\mathbf{12},13,k,\boldsymbol{j})$. Observe that j and k have no admissible values from V. On the other hand when $\mathrm{lk}(7) = C_7(3,12,8,\mathbf{1},6,11,\mathbf{2})$ then $\mathrm{lk}(12) = C_7(8,7,3,\mathbf{4},13,i,\boldsymbol{j})$, where $(i,j) \in \{(10,11),(14,9),(14,10)\}$. If $(i,j) = (10,11)$, successively considering lk(12), lk(10), lk(11), we see that $\deg(6) > 5$. If $(i,j) = (14,9)$, $\mathrm{lk}(9) = C_7(10,k,14,\mathbf{12},8,1,\mathbf{2})$ and we get no value for k from V. If $(i,j) = (14,10)$, $\mathrm{lk}(8) = C_7(1,9,10,\mathbf{14},12,7,\mathbf{6})$, this implies $C_4(1,2,10,8) \subseteq \mathrm{lk}(9)$. So $(g,h) \neq (12,13)$

Subcase 3.3. When $(e,f) = (11,8)$ then $\mathrm{lk}(8) = C_7(1,9,3,\mathbf{2},11,7,\mathbf{6})$ or $\mathrm{lk}(8) = C_7(1,9,11,\mathbf{2},3,7,\mathbf{6})$. In case, $\mathrm{lk}(8) = C_7(1,9,11,\mathbf{2},3,7,\mathbf{6})$, we have $\mathrm{lk}(9) = C_7(11,8,1,\mathbf{2},10,i,\boldsymbol{j})$, for $(i,j) \in \{(4,5),(5,4),(12,13)\}$. If $(i,j) = (4,5)$, $\mathrm{lk}(11) = C_7(2,10,5,\mathbf{4},9,8,\mathbf{3})$, now considering lk(10), we see that the set $\{4,5\}$ is an edge and non-edge both. If $(i,j) = (5,4)$, $\mathrm{lk}(11) = C_7(2,10,4,\mathbf{5},9,8,\mathbf{3})$, now considering lk(10) we see that the set $\{4,5\}$ is an edge and non-edge both. If $(i,j) = (12,13)$, $\mathrm{lk}(11) = C_7(2,10,13,\mathbf{12},9,8,\mathbf{3})$, now considering lk(10), we see that the set $\{12,13\}$ is an edge and non-edge both. On the other hand when $\mathrm{lk}(8) = C_7(1,9,3,\mathbf{2},11,7,\mathbf{6})$ then $\mathrm{lk}(9) = C_7(1,8,3,\mathbf{4},12,10,\mathbf{2})$, $\mathrm{lk}(3) = C_7(2,0,4,\mathbf{12},9,8,\mathbf{11})$. This implies $\mathrm{lk}(4) = C_7(5,0,3,\mathbf{9},12,g,\boldsymbol{h})$, where $(g,h) \in \{(7,11),(13,14)\}$. If $(g,h) = (13,14)$, $\mathrm{lk}(5) = C_7(6,0,4,\mathbf{13},14,i,\boldsymbol{j})$ and $\mathrm{lk}(6) = C_7(5,0,1,\mathbf{8},7,j,\boldsymbol{i})$, but we see that i and j have no admissible value from V. If $(g,h) = (7,11)$, $\mathrm{lk}(7) = C_7(4,12,6,\mathbf{1},8,11,\mathbf{5})$, this implies $\mathrm{lk}(6) = C_7(5,0,1,\mathbf{8},7,12,\boldsymbol{i})$. Observe that $i = 10$, now completing successively we get $\mathrm{lk}(6) = C_7(1,0,5,\mathbf{10},12,7,\mathbf{8})$, $\mathrm{lk}(5) = C_7(4,0,6,\mathbf{12},10,11,\mathbf{7})$, $\mathrm{lk}(10) = C_7(2,11,5,\mathbf{6},12,9,\mathbf{1})$, $\mathrm{lk}(11) = C_7(2,10,5,\mathbf{4},7,8,\mathbf{3})$, $\mathrm{lk}(12) = C_7(4,7,6,\mathbf{5},10,9,\mathbf{3})$. This gives $M \cong K_2(3^6 : 3^2,4,3,4)$ via $0 \mapsto u$, $1 \mapsto v_4$, $2 \mapsto v_3$, $3 \mapsto v_2$, $4 \mapsto v_1$, $5 \mapsto v_6$, $6 \mapsto v_5$, $7 \mapsto v_8$, $8 \mapsto v_{11}$, $9 \mapsto v_{10}$, $10 \mapsto v_{12}$, $11 \mapsto v_7$, $12 \mapsto v_9$.

Subcase 3.4. When $(e,f) = (11,12)$ then $\mathrm{lk}(3) = C_7(4,0,2,\mathbf{11},12,g,\boldsymbol{h})$. It is easy to see that $(g,h) \in \{(7,13),(8,9),(9,8),(13,7),(13,8),(13,9),(13,10),(13,14)\}$.

If $(g,h) = (7,13)$, then $\mathrm{lk}(7) = C_7(6,12,3,\mathbf{4},13,8,\mathbf{1})$ or $\mathrm{lk}(7) = C_7(8,12,3,\mathbf{4},13,6,\mathbf{1})$. In case $\mathrm{lk}(7) = C_7(6,12,3,\mathbf{4},13,8,\mathbf{1})$, $\mathrm{lk}(6) = C_7(5,0,1,\mathbf{8},7,12,\boldsymbol{i})$. Observe that $i \in \{9,10,13\}$.

If $i = 9$, $\text{lk}(9) = C_7(1, 8, 5, \mathbf{6}, 12, 10, \mathbf{2})$ or $\text{lk}(9) = C_7(1, 8, 12, \mathbf{6}, 5, 10, \mathbf{2})$, but for both the cases of lk(9), we see $\deg(12) > 5$. If $i = 10$ then $\text{lk}(12) = C_7(3, 7, 6, \mathbf{5}, 10, 11, \mathbf{2})$ and we get $C_4(2, 3, 12, 10) \subseteq \text{lk}(11)$. If $i = 13$, $\text{lk}(12) = C_7(3, 7, 6, \mathbf{5}, 13, 11, \mathbf{2})$ and we get $\deg(13) > 5$. Thus $\text{lk}(7) \neq C_7(6, 12, 3, \mathbf{4}, 13, 8, \mathbf{1})$ On the other hand when $\text{lk}(7) = C_7(8, 12, 3, \mathbf{4}, 13, 6, \mathbf{1})$ then $\text{lk}(6) = C_7(5, 0, 1, \mathbf{8}, 7, 13, \boldsymbol{i})$. Observe that $i \in \{9, 10, 11, 14\}$. If $i = 9$, considering successively lk(13), lk(10), lk(5), we get $\deg(9) > 5$. If $i = 10$, $\text{lk}(13) = C_7(4, j, 10, \mathbf{5}, 6, 7, \mathbf{3})$, where $j \in \{9, 11\}$. If $j = 9$, considering successively lk(10), lk(9) and lk(5), we get $\deg(10) > 5$ and if $j = 11$, we get three consecutive triangular faces incident at 11. When $i = 11$, three consecutive triangular faces at 10. When $i = 14$, $\text{lk}(13) = C_7(4, j, 14, \mathbf{5}, 6, 7, \mathbf{3})$ and we get no value for j in V. So $(g, h) \neq (7, 13)$

If $(g, h) = (8, 9)$, successively we get $\text{lk}(8) = C_7(3, 12, 7, \mathbf{6}, 1, 9, \mathbf{4})$, $\text{lk}(9) = C_7(4, 13, 10, \mathbf{2}, 1, 8, \mathbf{3})$. This implies $\text{lk}(4) = C_7(5, 0, 3, \mathbf{8}, 9, 13, \boldsymbol{j})$, where $i \in \{7, 11, 14\}$. When $i = 7$, $\text{lk}(7) = C_7(5, 12, 8, \mathbf{1}, 6, 13, \mathbf{4})$, now considering lk(4) and lk(6), we see that two distinct quadrangular faces share more than one vertex. When $i = 11$ then successively we get $\text{lk}(11) = C_7(5, 10, 2, \mathbf{3}, 12, 13, \mathbf{4})$, $\text{lk}(10) = C_7(2, 11, 5, \mathbf{6}, 13, 9, \mathbf{1})$. But $\deg(13) > 5$. If $i = 14$, successively, we get $\text{lk}(13) = C_7(4, 9, 10, \mathbf{12}, 7, 14, \mathbf{5})$, $\text{lk}(7) = C_7(6, 14, 13, \mathbf{10}, 12, 8, \mathbf{1})$ and $\text{lk}(6) = C_7(5, 0, 1, \mathbf{8}, 7, 14, \boldsymbol{j})$, now observe that the set $\{5, 14\}$ forms an edge and non-edge both. So, $(g, h) \neq (8, 9)$

Computing similarly for $(g, h) \in \{(8, 9), (9, 8), (13, 7), (13, 8), (13, 9), (13, 10), (13, 14)\}$, one can see easily that no map exists and therefore $(e, f) \neq (11, 12)$. This completes the exhaustive search and thus the proof. □

§5. Discussion

Note that ASEMs are a generalization of maps on Johnson solids to the close surfaces other than the 2-sphere. One can construct infinitely many types ASEMs on the sphere as follows: consider an n-gonal disk D_n, $n \geq 4$, with vertex set $V(D_n) = \{a_1, a_2, \cdots, a_n\}$ and a vertex a out side this disk, now join a to each a_i, $1 \leq i \leq n$, by an edge. This gives ASEMs of type $[(3.3.\ldots.3(n\text{-times}))_1 : (3.3.n)]$ for each $n \geq 4$. In the present work, existence of ASEMs is shown for the surfaces of Euler characteristic 0, that is, for the torus and Klein bottle. This study motivates us to explore other types of ASEMs on the torus and Klein bottle including other closed surfaces. As a consequence, the following natural question occurs.

Question 5.1 *Can we construct ASEMs of types, other than* $[(3, 3, 3, 3, 3, 3)_1 : (3, 3, 4, 3, 4)]$, *on the torus or Klein bottle?*

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Refernces

[1] U. Brehm, Polyhedral maps with few edges, *Topics in Combinatorics and Graph Theory*,

Physica-Verlag: Heidelberg, 1990.

[2] U. Brehm, B. Datta and N. Nilakantan, The edge-minimal polyhedral maps of Euler characteristic -8, *Beitr. Algebra Geom.*, 43(2001), 583-596.

[3] U. Brehm, W. Kuhnel, Equivelar maps on the torus, *European J. Combin.*, 29(2008), 1843-1861.

[4] B. Datta and N. Nilakantan, Equivelar Polyhedra with few vertices, *Discrete Comput. Geom.*, 26(2001), 429-461.

[5] B. Datta and A.K. Upadhyay, Degree-regular triangulations of torus and Klein bottle, *Proc. Indian Acad. Sci. Math. Sci.*, 115(2005), 279-307.

[6] B. Datta and A.K. Upadhyay, Degree-regular triangulations of the double torus, *Forum Math.*, 18(2006), 1011-1025.

[7] J.L. Gross and T.W. Tucker, *Topological Graph Theory*, John Wiley and Sons, New York, 1987.

[8] N.W. Johnson. Convex solids with regular faces, *Canadian J. Math.*, 18 (1966), 169-200.

[9] J. Karabas and R. Nedela, Archimedean solids of genus 2, *Electron. Notes Disc. Math.*, 28(2007), 331-339.

[10] J. Karabas and R. Nedela, Archimedean maps of higher genera, *Math. Comp.*, 81(2012), 569-583.

[11] A.K. Tiwari and A.K. Upadhyay, Semi-equivelar maps on the surfaces of Euler characteristic -1, *Note Mat.*, 37(2017), 91-102.

[12] A.K. Tiwari and A.K. Upadhyay, Semi-equivelar maps on the torus and the Klein bottle with few vertices, *Math. Slovaca*, 67(2017), 519-532.

[13] A.K. Upadhyay, A.K. Tiwari and D. Maity, Semi-equivelar maps. *Beitr. Algebra Geom.*, 55(2012), 229-242.